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Simultaneous Orthogonal Rotations Angle

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Abstract. Angular orientation refers to the position of a rigid body intrinsic coordinate system relative to a reference coordinate system with the same origin. It is determined with a sequence of rotations needed to move the rigid-body coordinate-system axes initially aligned with the reference coordinate-system axes to their new position. In this paper we present a novel way for representing angular orientation. We define the Simultaneous Orthogonal Rotations Angle (SORA) vector with components equal to the angles of three simultaneous rotations around the coordinate-system axes. The problem of non-commutativity is here avoided. We numerically verify that SORA is equal to the rotation vector – the three simultaneous rotations it comprises are equivalent to a single rotation. The axis of this rotation coincides with the SORA vector while the rotation angle is equal to its magnitude. We further verify that if the coordinate systems are initially aligned, simultaneous rotations around the reference and rigid-body intrinsic axes represent the same angular orientation. Considering the SORA vector, angular orientation of a rigid body can be calculated in a single step thus avoiding the iterative infinitesimal rotation approximation computation. SORA can thus be a very convenient way for angular-orientation representation.

Keywords: angular orientation, rotation, rotation axis, rotation angle

1 INTRODUCTION

Angular orientation refers to the position of a rigid-body intrinsic coordinate system relative to a reference coordinate system with the same origin. It is determined with a rotation needed to move the rigid-body coordinate system initially aligned with the reference coordinate system to its new position.

The reference and the rigid-body intrinsic coordinate system are considered Cartesian with axes x, y, z and x', y', z', respectively. Orientation of the system axes conforms to the right hand rule. Both coordinate systems are illustrated in Fig. 1.

According to the Euler's rotation theorem [1 p.83] any two independent coordinate systems of the same origin can be related by a sequence of not more than three rotations around the coordinate axes, where no two successive rotations may be about the same axis. The successive rotation axis distinction restriction from the Euler's theorem permits twelve different rotation axes sequences.

Angular orientation can be specified as a sequence of rotations around the reference or rigid-body intrinsic coordinate-system axes. This consideration doubles the number of possible rotation sequences. As rotation sequences are not commutative, each of these 24 rotation sequences in general specifies different angular orientations.

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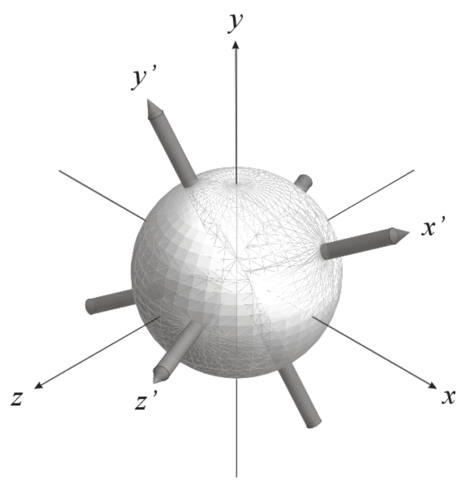

Figure 1: Reference and rigid-body intrinsic coordinate systems.

In this paper we present angular orientation with three simultaneous rotations around the coordinate-system axes. We define the Simultaneous Orthogonal Rotations Angle (SORA) vector with components equal to the angles of these three simultaneous rotations, thus avoiding the problem of non-commutativity. SORA represents a unique position of a rigid-body intrinsic coordinate system relative to a reference coordinate system with the same origin.

We numerically verify that SORA is equal to the rotation vector – the three simultaneous rotations of the SORA notation are equivalent to a single rotation. The axis of this rotation coincides with the SORA vector while the rotation angle is equal to its magnitude. To allow for verification, we compare matrix representations of angular orientation obtained by using SORA and by considering the total rotation being a consequence of a repeating sequence of approximately infinitesimal rotations around three coordinate axes. The order of applying infinitesimal rotations is not important, as these are shown to be commutative [2].

2 ANGULAR ORIENTATION REPRESENTATION

Said above, angular orientation is determined with a rotation needed to move a rigid-body coordinate system initially aligned with the reference coordinate system to its new position. Many different notations are used to represent rotation. These include Euler angles, rotation matrix, axis and angle, rotation vector, and quaternions. There exist also some other notations like the Rodrigues and Cayley-Klein parameter, but are not widely used.

2.1 Euler angles

The three successive rotation angles described in the Euler's rotation theorem are called Euler angles [1 p.83, 3, 4]. If the rigid-body intrinsic and the reference coordinate systems are initially aligned, the Euler angles sequence may be identified by specifying the axes of rotation of each of the three rotations. These are expressed in terms of the reference coordinate system and the two intermediate coordinate systems [3]. The orientations of these two intermediate coordinate-system axes are specified with the first and the second successive rotation. The first rotation may be about any of the three orthogonal axes of the reference coordinate system. Considering the successive rotation-axis distinction restriction, the second and third rotation may be about either of the two axes of the first and second intermediate coordinate system. This gives twelve Euler-angle sequence possibilities. Each of these twelve sequences is equivalent to one of the possible twelve rotation sequences around the reference coordinate-system axes. This gives a total of 24 different notations.

2.2 Axis and angle

The Euler's rotation theorem also states that any rotation about the three coordinate-system axes can be expressed as a single rotation about some new axis. The axis and angle [5, 6] is a pair comprising a unit vector representing a rotation axis and an angle of rotation around that axis:

$$(\mathbf{a}_\mathrm{r},\varphi)=\left(\begin{bmatrix}a_x\\a_y\\a_z\end{bmatrix},\varphi\right) \tag{1}$$

2.3 Rotation vector

The rotation axis and angle can be comprised in a single non-normalized rotation vector which has the same direction as the rotation axis while its magnitude is equal to the rotational angle:

$$\mathbf{r}=\mathbf{a}_\mathrm{r}\cdot\varphi=\begin{bmatrix}a_x\cdot\varphi\\a_y\cdot\varphi\\a_z\cdot\varphi\end{bmatrix} \tag{2}$$

It is important to note that successive rotations cannot be represented by rotation vector addition.

2.4 Rotation matrix

Angular orientation of the rigid-body coordinate system can be completely specified with a system of three base vector equations. Each base vector of the rigid-body intrinsic coordinate system is expressed as a linear combination of the reference base vectors:

$$\begin{aligned}\mathbf{u}_x{}'&=r_{xx}\cdot\mathbf{u}_x+r_{xy}\cdot\mathbf{u}_y+r_{xz}\cdot\mathbf{u}_z\\\mathbf{u}_y{}'&=r_{yx}\cdot\mathbf{u}_x+r_{yy}\cdot\mathbf{u}_y+r_{yz}\cdot\mathbf{u}_z\\\mathbf{u}_z{}'&=r_{zx}\cdot\mathbf{u}_x+r_{zy}\cdot\mathbf{u}_y+r_{zz}\cdot\mathbf{u}_z\end{aligned} \tag{3}$$

Nine parameters from equations (3) form a 3x3 rotation matrix [7]:

$$\mathbf{R}=\begin{bmatrix}r_{xx}&r_{yx}&r_{zx}\\r_{xy}&r_{yy}&r_{zy}\\r_{xz}&r_{yz}&r_{zz}\end{bmatrix} \tag{4}$$

Matrix (4) represents a rotation needed to move the rigid-body intrinsic coordinate system initially aligned with the reference coordinate system to its new position. The rotation matrix (4) is real and orthonormal:

$$\mathbf{R}\cdot\mathbf{R}^T=\mathbf{R}^T\cdot\mathbf{R}=1 \tag{5}$$

having a determinant equal to:

$$|\mathbf{R}|=1 \tag{6}$$

A product of two or more rotation matrices is again a rotation matrix, representing a rotation sequence, where the rotation order in a sequence is important.

2.5 Quaternion

Quaternions [1 p.106-112] form a four-dimensional vector space with base elements denoted with 1, i, j, and k:

$$q = a + i \cdot b + j \cdot c + k \cdot d \tag{7}$$

where:

$$i^2 = j^2 = k^2 = i \cdot j \cdot k = -1 \tag{8}$$

In expression (7), a refers to quaternion q real part and $b, c,$ and d to its imaginary parts. A thorough insight into different quaternion operations can be found in [1 p.106-112]. It has been shown that a normalized quaternion:

$$a^2 + b^2 + c^2 + d^2 = 1 \tag{9}$$

represents a rotation of angle φ around an oriented vector $\begin{bmatrix} x_0 & y_0 & z_0 \end{bmatrix}$. The rotation angle and axis are defined with the components of the normalized quaternion:

$$q = \cos\frac{\varphi}{2} + i \cdot \sin\frac{\varphi}{2} + j \cdot \sin\frac{\varphi}{2} + k \cdot \sin\frac{\varphi}{2} \tag{10}$$

Rotation of arbitrary vector $\mathbf{v}$ is expressed as a quaternion product:

$$v_R = q \cdot v \cdot q^* \tag{11}$$

where the oriented rotation vector and angle are given with quaternion q components (10). q^* denotes the complex conjugate of q. Vectors $\mathbf{v}$ and $\mathbf{v_R}$ in equation (11) appear as quaternions with real parts equal to 0.

Angular orientation of the rigid-body intrinsic coordinate-system axes using quaternions is then represented according to the following:

$$\begin{aligned} qu_x' &= q \cdot qu_x \cdot q^* \\ qu_y' &= q \cdot qu_y \cdot q^* \\ qu_z' &= q \cdot qu_z \cdot q^* \end{aligned} \tag{12}$$

where imaginary parts of quaternions $qu_x, qu_y, qu_z, qu_x', qu_y',$ and qu_z' are equal to the respective coordinate-system base vectors while their real part is equal to zero.

A product of the quaternions representing rotations is also a quaternion representing the sequence of these rotations where, as is the case with the rotation matrices, the rotation order in a sequence is important.

3 Simultaneous Orthogonal Rotations Angle

3.1 Definition

In the chapter above, our focus was on sequential rotations enabling us to represent angular orientation of a rigid-body coordinate system. Let us now represent angular orientation with constant simultaneous rotations. Let $\omega_x, \omega_y,$ and ω_z represent angular velocities of the rigid-body intrinsic coordinate axis rotating around the reference axes and let further $\phi_x, \phi_y,$ and ϕ_z represent the respective angular displacements. We can then write:

$$\omega_x = \frac{d\phi_x}{dt};\ \omega_y = \frac{d\phi_y}{dt};\ \omega_z = \frac{d\phi_z}{dt}. \tag{13}$$

Let us further consider angular velocities as vectors:

$$\mathbf{\Omega}_x = \begin{bmatrix} \omega_x \\ 0 \\ 0 \end{bmatrix};\ \mathbf{\Omega}_y = \begin{bmatrix} 0 \\ \omega_y \\ 0 \end{bmatrix};\ \mathbf{\Omega}_z = \begin{bmatrix} 0 \\ 0 \\ \omega_z \end{bmatrix} \tag{14}$$

Angular velocity vectors (14) are aligned with rotation axes and their magnitudes are equal to angular velocities. Their sum is a 3D angular velocity vector:

$$\mathbf{\Omega} = \mathbf{\Omega}_x + \mathbf{\Omega}_y + \mathbf{\Omega}_z = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{15}$$

Considering (13)-(15) we define SORA, as a new notation used in representing angular orientation:

$$\mathbf{\Phi} = \mathbf{\Omega} \cdot t = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \cdot t = \begin{bmatrix} \phi_x \\ \phi_y \\ \phi_z \end{bmatrix} \tag{16}$$

where t is the total time of rotation. The components of the SORA vector (16) are the angles of the three simultaneous rotations around the coordinate-system axes.

According to the Euler's rotational theorem, as mentioned above, any rotation about the three coordinate-system axes can be expressed as a single rotation about some new axis. Let us now assume that the three simultaneous rotations comprised in the SORA vector can be considered as one single rotation, the axis and angle of which are equal to the SORA vector orientation and magnitude, respectively:

$$\mathbf{u} = \frac{\mathbf{\Phi}}{\|\mathbf{\Phi}\|} = \frac{\mathbf{\Phi}}{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}} \tag{17}$$

$$\varphi = \|\mathbf{\Phi}\| = \sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2} \tag{18}$$

For the time being we are unable to prove this assumption. However, we can verify it numerically. Our verification is presented in the next section.

3.2 Numerical verification

Finite rotation denoted with rotation matrix $\mathbf{R}$ can be represented as an infinite sequence of infinitesimal rotations denoted with rotation matrix $d\mathbf{R}$. Let us consider this representation and approximate the three simultaneous rotations around the reference coordinate-system axes described in the previous section with n consecutive sequences of rotations around the coordinate-system axes for small angles $\Delta\phi_x$, $\Delta\phi_y$, and $\Delta\phi_z$ where

$$\Delta\phi_x = \frac{\phi_x}{n};\ \Delta\phi_y = \frac{\phi_y}{n};\ \Delta\phi_z = \frac{\phi_z}{n} \tag{19}$$

Each of these rotation sequences can be represented with rotation matrix $\mathbf{\Delta R}_{xyz}$:

$$\mathbf{\Delta R}_{xyz} = \mathbf{R}(\Delta\phi_z, \mathbf{u}_z) \cdot \mathbf{R}(\Delta\phi_y, \mathbf{u}_y) \cdot \mathbf{R}(\Delta\phi_x, \mathbf{u}_x) \tag{20}$$

where rotation matrices $\mathbf{R}(\Delta\phi_x, \mathbf{u}_x)$, $\mathbf{R}(\Delta\phi_y, \mathbf{u}_y)$, and $\mathbf{R}(\Delta\phi_z, \mathbf{u}_z)$ represent individual rotations around the coordinate-system axes.

To rotate the rigid-body intrinsic coordinate system for angles ϕ_x, ϕ_y, and ϕ_z the above sequence of small rotations must be repeated n times. The rotation matrix of the infinite sequence of infinitesimal rotations approximation $\mathbf{R}_{xyz}(n)$ is thus given as the n^{th} power of rotation matrix $\mathbf{\Delta R}_{xyz}$:

$$\mathbf{R}_{xyz}(n) = \mathbf{\Delta R}_{xyz}{}^{n} \tag{21}$$

We can obtain any other of the 12 possible sequences simply by changing the matrix multiplication order in the above procedure. As infinitesimal rotations are commutative [2], we expect, for large n, to obtain the same result with different rotation sequences.

The above rotation matrix represents rotation around the reference coordinate-system axes. If the rigid body rotates around its intrinsic coordinate-system axes, the corresponding approximation rotation matrix denoted with $\mathbf{R}_{xyz}(n)'$ can be obtained by performing n steps of the following recursive equations:

$$\begin{aligned} \mathbf{P}_i &= \mathbf{R}(\Delta\phi_x, \mathbf{R}_{xyz}(n)'_i) \\ \mathbf{Q}_i &= R(\Delta\phi_y, \mathbf{P}_i \cdot \mathbf{u}_y) \\ \mathbf{R}_{xyz}(n)'_{i+1} &= \mathbf{R}(\Delta\phi_z, \mathbf{Q}_i \cdot \mathbf{u}_z) \end{aligned} \tag{22}$$

where the initial value of rotation matrix $\mathbf{R}_{xyz}(n)'$ is:

$$\mathbf{R}_{xyz}(n)'_0 = \mathbf{I} \tag{23}$$

and $\mathbf{I}$ is a 3×3 identity matrix.

Let us now say that our approximation is good enough provided the resulting rotation matrices for the two considered rotation sequences around the reference coordinate-system axes $\mathbf{R}_{xyz}(n)$ and $\mathbf{R}_{yzx}(n)$ match up to the 6^{th} decimal place. We calculated the rotation matrices according to the above procedure using Mathematica [8] for numerous examples. For all, the demanded precision was achieved when $n = 10^7$.

We further calculated rotation matrix $\mathbf{R}_{xyz}(n)'$ representing rotations around the rigid-body intrinsic coordinate axes when $n = 10^7$ for the same example set. Comparing $\mathbf{R}_{xyz}(n)'$ with respective $\mathbf{R}_{xyz}(n)$ and $\mathbf{R}_{yzx}(n)$ results, these also matched up to the 6 decimal place. This shows that rotations around the reference and rigid-body intrinsic coordinate-system axes give the same result.

Finally, we calculated rotation matrix $\mathbf{R}_{SORA}$ representing a single rotation around an axis defined with the SORA vector orientation (16) and for an angle defined with the SORA vector magnitude (17) as specified in the previous section. The $\mathbf{R}_{SORA}$ results matched all considered approximations of rotation matrices $\mathbf{R}_{xyz}(n)$, $\mathbf{R}_{yzx}(n)$, and $\mathbf{R}_{xyz}(n)'$ up to the demanded precision for all considered examples. This verifies that the SORA vector is equal to the rotation vector.

As said above, a number of numerical examples were considered. Here we present only an illustrational example where angular orientation of a rigid-body coordinate system is represented with angular velocity angles $\phi_x = 0.75$, $\phi_y = 0.90$, and $\phi_z = 0.60$. Our results obtained by using Mathematica [8] for $n = 10^7$ all matched up to the 6^{th} decimal place:

$$\mathbf{R}_{SORA} = \mathbf{R}_{xyz}(n) = \mathbf{R}_{yzx}(n) = \mathbf{R}_{xyz}(n)' = \\ = \begin{bmatrix} 0.494730 & -0.149651 & 0.856065 \\ 0.732655 & 0.601614 & -0.318240 \\ -0.467395 & 0.784643 & 0.407280 \end{bmatrix} \quad (24)$$

4 CONCLUSION

In this paper we defined the Simultaneous Orthogonal Rotations Angle (SORA) vector with components equal to the angles of three simultaneous orthogonal rotations around the rigid-body intrinsic coordinate axes.

We numerically verified that SORA is equal to the rotation vector – the three simultaneous rotations it comprises are equivalent to a single rotation around the SORA vector and for an angle equal to its magnitude.

We further showed that simultaneous rotations around the reference and rigid-body intrinsic coordinate-system axes represent the same angular orientation.

Considering the SORA vector, angular orientation of a rigid body can be calculated in a single step thus avoiding the iterative infinitesimal rotation approximation computation.

SORA can thus be a very convenient way for angular orientation representation.

In our further research, we will try to derive the presented results analytically.

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